HAUSDORFF DIMENSION OF CERTAIN RANDOM SELF–AFFINE FRACTALS

NUNO LUZIA
UFRJ, Rio de Janeiro, Brasil
e-mail address: luzia@impa.br

ABSTRACT

In this work we are interested in the self–affine fractals studied by Gatzouras and Lalley [5] and by the author [8] which generalize the famous *general Sierpiński carpets* studied by Bedford [2] and McMullen [10]. We give a formulae for the Hausdorff dimension of sets which are *randomly* generated using a finite number of self–affine transformations each one generating a fractal set as mentioned before, with some technical hypotheses. The choice of the transformation is random according to a Bernoulli measure. The formulae is given in terms of the variational principle for the dimension.

Keywords: Hausdorff dimension; random; variational principle

1. Introduction

A main difficulty in calculating Hausdorff dimension is the phenomenon of *non-conformality* which arises when we have several rates of expansion. In the 1-dimensional (conformal) setting the computation of Hausdorff dimension is possible, at least in the *uniformly expanding* context, due to the thermodynamic formalism introduced by Sinai-Ruelle-Bowen (see [3] and [12]). The problem of calculating Hausdorff dimension in the non-conformal setting was first considered by Bedford [2] and McMullen [10]. They showed independently that for the class of transformations called *general Sierpiński carpets*, there exists an ergodic measure of full Hausdorff dimension. Following these works several extensions have been made, e.g. in [1], [5], [6], [7], [8] and [9]. In this work we are interested in the self–affine fractals studied in [5] and [8], which we call *self–affine Sierpiński carpets*, namely we compute the Hausdorff dimension of a random version of these sets, with some technical hypotheses.

Let $m \in \mathbb{N}$ and consider the symbolic space $\mathcal{I} = \{1, ..., m\}^{\mathbb{N}}$ equipped with a Bernoulli measure given by the probability vector $\mathbf{p} = (p_1, ..., p_m)$ (we assume $p_i > 0$ for all i). This will be our space to which *random* refers to.

For the moment let us fix $i \in \{1, ..., m\}$ and remember the construction of *self–affine Sierpiński carpets*. Let $S_1, S_2, ..., S_r$ be contractions of $\mathbb{R}^2$. Then there is a unique nonempty compact set Λ of $\mathbb{R}^2$ such that

$$\Lambda = \bigcup_{l=1}^{r} S_l(\Lambda).$$

We will refer to Λ as the limit set of the semigroup generated by $S_1, S_2, ..., S_r$. Now consider the sets Λ which are limit sets of the semigroup generated by the

2-dimensional mappings A_{ijk} given by

$$A_{ijk} = \begin{pmatrix} a_{ijk} & 0 \\ 0 & b_{ij} \end{pmatrix} x + \begin{pmatrix} c_{ijk} \\ d_{ij} \end{pmatrix}$$

for $(j,k) \in \mathcal{J}_i$, where

$$\mathcal{J}_i = \{(j,k) : \ 1 \le j \le m_i, \ 1 \le k \le m_{ij}\}$$

is a finite index set. We assume $0 < a_{ijk} \le b_{ij} < 1$, $\sum_{j=1}^{m_i} b_{ij} \le 1$ and $\sum_{k=1}^{m_{ij}} a_{ijk} \le 1$. Also, $0 \le d_{i1} < d_{i2} < ... < d_{im_i} < 1$ with $d_{ij+1} - d_{ij} \ge b_{ij}$ and $1 - d_{im_i} \ge b_{im_i}$, and $c_{ij1} < c_{ij2} < ... < c_{ijm_{ij}} < 1$ with $c_{ij(k+1)} - c_{ijk} \ge a_{ijk}$ and $1 - c_{ijm_{ij}} \ge a_{ijm_{ij}}$. These hypotheses guarantee that the rectangles

$$R_{ijk} = A_{ijk}([0,1]^2)$$

have interiors that are pairwise disjoint, with edges parallel to the x- and y-axes, are arranged in "rows" of height b_{ij} and have width a_{ijk}. Geometrically, Λ is the limit (in the Hausdorff metric), or the intersection, of *n-approximations*: the 1-approximation consisting of the rectangles R_{ijk}, the 2-approximation consisting in replacing each rectangle of the 1-approximation by an affine copy of the 2-approximation, and so on. Formally,

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(j_1,k_1),\dots,(j_n,k_n) \in \mathcal{J}_i} A_{ij_1k_1} \circ \cdots \circ A_{ij_nk_n}([0,1]^2).$$

Now we want to give a random version of this construction in such a way that at each step of the approximation we are allowed to change the number $i \in \{1, ..., m\}$. More precisely, given $\mathbf{i} = (i_1, i_2, ...) \in \mathcal{I}$, we consider the *random set* given by

$$\Lambda_{\mathbf{i}} = \bigcap_{n=1}^{\infty} \bigcup_{(j_1,k_1) \in \mathcal{J}_{i_1},\dots,(j_n,k_n) \in \mathcal{J}_{i_n}} A_{i_1j_1k_1} \circ \cdots \circ A_{i_nj_nk_n}([0,1]^2).$$

Of course this construction generalizes the previous one by putting $\mathbf{i} = (i, i, ...)$.

We will need the following *generic* hypothesis on the numbers a_{ijk}. For each $t \in [0,1]$, there exist $1 \le i \le m$ and $1 \le j < j' \le m_i$ such that

$$\sum_{k=1}^{m_{ij}} a_{ijk}^t \ne \sum_{k=1}^{m_{ij'}} a_{ij'k}^t. \tag{1}$$

We will also need one of the following *robust* hypotheses. Let $\varepsilon > 0$. For each $i \in \{1, ..., m\}$, there exist numbers $0 < a_i \le b_i < 1$ such that

$$(1+\varepsilon)^{-1} < \frac{b_{ij}}{b_i} < 1 + \varepsilon, \quad (1+\varepsilon)^{-1} < \frac{a_{ijk}}{a_i} < 1 + \varepsilon, \tag{2}$$

or

$$\frac{b_{ij}}{a_{ijk}} < 1 + \varepsilon, \tag{3}$$

for all $(j,k) \in \mathcal{J}_i$.

Notation: $\dim_{\mathrm{H}} \Lambda$ stands for the Hausdorff dimension of a set Λ.

Theorem A. *There exists $\varepsilon > 0$ such that if (1) and (2) or (1) and (3) are satisfied then*

$$\dim_{\mathrm{H}} \Lambda_{\mathbf{i}} = \sup_{\mathbf{P}} \{\lambda(\mathbf{P}) + t(\mathbf{P})\} \quad \textit{for } \mathbf{p}\textit{-a.e. } \mathbf{i}. \tag{4}$$

where $\mathbf{P} = (p_{ij})$ is a collection of non-negative numbers satisfying

$$\sum_{j=1}^{m_i} p_{ij} = p_i, \quad i = 1, ..., m,$$

the number $\lambda(\mathbf{P})$ *is given by*

$$\lambda(\mathbf{P}) = \frac{\sum_{i,j} p_{ij} \log p_{ij} - \sum_i p_i \log p_i}{\sum_{i,j} p_{ij} \log b_{ij}},$$

(by convention: $0 \log 0 = 0$*)*
and $t(\mathbf{P})$ *is the unique real in* $[0,1]$ *satisfying*

$$\sum_{i,j} p_{ij} \log \left(\sum_k a_{ijk}^{t(\mathbf{P})} \right) = 0.$$

Remark 1. The number $\lambda(\mathbf{P})$ is the Hausdorff dimension in y-axis of the set of generic points for the distribution $\mathbf{P}|\mathbf{p}$; the number $t(\mathbf{P})$ is the Hausdorff dimension of a typical 1-dimensional fibre in the x-direction relative to the distribution $\mathbf{P}$, and is given by a random Moran formula. Also note that by putting $p_i = 1$ for some i we get the deterministic case, i.e. the Hausdorff dimension of self-affine Sierpiński carpets (satisfying the technical hypotheses).

It follows from the proof of Theorem A (see Lemma 1) that the expression between brackets in (4) is the Hausdorff dimension of a Bernoulli measure $\mu_{\mathbf{P}}$. Since the functions $\mathbf{P} \mapsto \lambda(\mathbf{P})$ and $\mathbf{P} \mapsto t(\mathbf{P})$ are continuous, we obtain the following.

Corollary A. *With the same hypotheses of Theorem A, there exists* $\mathbf{P}^*$ *such that*

$$\dim_{\mathrm{H}} \Lambda_{\mathbf{i}} = \dim_{\mathrm{H}} \mu_{\mathbf{P}^*} \quad \textit{for } \mathbf{p}\textit{-a.e. } \mathbf{i}.$$

Example 1. Let k be a positive integer and $0 < q < 1$. Divide the unit square into a grid of $k \times k$ squares with side length k. Consider the random set Λ_q constructed as before as the limit of n-approximations, such that at each step of a further approximation we use a transformation that corresponds to a grid where each square has probability q to belong to this grid. This construction differs from the usual *fractal percolation* in that the randomness is with respect to the grid (the transformation) and not on each square. This construction is a particular case of ours if one considers all the possible patterns for the selected squares of the grids. Namely, the grids consisting of $1 \le l \le k^2$ squares are in the number of $\binom{k^2}{l}$ and we assign to each of these grids the probability $aq^l(1-q)^{k^2-l}$, where $a = (1-(1-q)^{k^2})^{-1}$ appears because we are excluding the case which no square is selected. Then,

$$a \sum_{l=1}^{k^2} \binom{k^2}{l} q^l (1-q)^{k^2-l} = 1.$$

Theorem A says that

$$\dim_{\mathrm{H}} \Lambda_q = \frac{1}{\log k} \sup_{\mathbf{P}} \Big\{ -\sum_{i,j} p_{ij} \log p_{ij} + \sum_i p_i \log p_i + \sum_{i,j} p_{ij} \log m_{ij} \Big\} \quad \text{a.s.}$$

Using Lagrange multipliers with the restrictions $\sum_j p_{ij} = p_i$ we get that the supremum above is attained at the probability vector

$$p_{ij} = p_i \frac{m_{ij}}{\sum_j m_{ij}}$$

and

$$\dim_{\mathrm{H}} \Lambda_q = \frac{\sum_i p_i \log\Big(\sum_j m_{ij}\Big)}{\log k} = \frac{a \sum_{l=1}^{k^2} \binom{k^2}{l} q^l (1-q)^{k^2-l} \log l}{\log k} \quad \text{a.s.}$$

Note that in the above formula $\log k$ corresponds to the Lyapunov exponent and $\log l$ corresponds to the topological entropy relative to a grid with l squares, so the numerator is the average of the entropies of the transformations. In this work we

are interested in the Hausdorff dimension of random fractals in a non-conformal setting and parameterized by real numbers.

2. Basic results

Here we mention some basic results about fractal geometry and pointwise dimension. For proofs we refer the reader to the books [4] and [11].

We are going to define the Hausdorff dimension of a set $F \subset \mathbb{R}^n$. The diameter of a set $U \subset \mathbb{R}^n$ is denoted by $|U|$. If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover F, i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ with $|U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of F. Given $t \geq 0$, we define the *t-dimensional Hausdorff measure of* F as

$$\mathcal{H}^t(F) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^t : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

It is not difficult to see that there is a critical value t_0 such that

$$\mathcal{H}^t(F) = \begin{cases} \infty & \text{if } t < t_0 \\ 0 & \text{if } t > t_0. \end{cases}$$

We define the *Hausdorff dimension* of F, written $\dim_{\mathrm{H}} F$, as being this critical value t_0.

Let μ be a Borel probability measure on $\mathbb{R}^n$. The Hausdorff dimension of the measure μ was defined by L.-S. Young as

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} F : \mu(F) = 1\}.$$

So, by definition, one has

$$\dim_{\mathrm{H}} F \geq \sup\{\dim_{\mathrm{H}} \mu : \mu(F) = 1\}.$$

In this paper we are interested in the validity of the opposite inequality in a dynamical context. In practice, to calculate the Hausdorff dimension of a measure, it is useful to compute its *lower pointwise dimension*:

$$\underline{d}_{\mu}(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ stands for the open ball of radius r centered at the point x. The relations between these dimensions are given by the following propositions.

Proposition 1.

(1) *If* $\underline{d}_{\mu}(x) \geq d$ *for* μ*-a.e.* x *then* $\dim_{\mathrm{H}} \mu \geq d$.
(2) *If* $\underline{d}_{\mu}(x) \leq d$ *for* μ*-a.e.* x *then* $\dim_{\mathrm{H}} \mu \leq d$.
(3) *If* $\underline{d}_{\mu}(x) = d$ *for* μ*-a.e.* x *then* $\dim_{\mathrm{H}} \mu = d$.

Proposition 2. *If* $\underline{d}_{\mu}(x) \leq d$ *for every* $x \in F$ *then* $\dim_{\mathrm{H}} F \leq d$.

3. Proof of Theorem A

Part 1: $\dim_{\mathrm{H}} \Lambda_{\mathbf{i}} \geq \sup_{\mathbf{P}}\{\lambda(\mathbf{P}) + t(\mathbf{P})\}$

There is a natural symbolic representation associated with our system that we shall describe now. Given $\mathbf{i} = (i_1, i_2, ...)$, consider the sequence space $\Omega_{\mathbf{i}} = \prod_{n=1}^{\infty} \mathcal{J}_{i_n}$. Elements of $\Omega_{\mathbf{i}}$ will be represented by $\omega = (\omega_1, \omega_2, ...)$ where $\omega_n = (j_n, k_n) \in \mathcal{J}_{i_n}$. Given $\omega \in \Omega$ and $n \in \mathbb{N}$, let $\omega(n) = (\omega_1, \omega_2, ..., \omega_n)$ and define the *cylinder of order* n,

$$C^{\mathbf{i}}_{\omega(n)} = \{\omega' \in \Omega_{\mathbf{i}} : \omega'_l = \omega_l,\, l = 1, ..., n\},$$

and the *basic rectangle of order* n,

$$R^{\mathbf{i}}_{\omega(n)} = A_{i_1\omega_1} \circ A_{i_2\omega_2} \circ \cdots \circ A_{i_n\omega_n}([0,1]^2).$$

We have that $(R^{\mathbf{i}}_{\omega(n)})_n$ is a decreasing sequence of closed rectangles having edges with length $\prod_{l=1}^n b_{i_l j_l}$ and $\prod_{l=1}^n a_{i_l j_l k_l}$. Thus $\bigcap_{n=1}^\infty R^{\mathbf{i}}_{\omega(n)}$ consists of a single point which belongs to $\Lambda_{\mathbf{i}}$ that we denote by $\chi_{\mathbf{i}}(\omega)$. This defines a continuous and surjective map $\chi_{\mathbf{i}}\colon \Omega_{\mathbf{i}} \to \Lambda_{\mathbf{i}}$ which is at most 4 to 1, and only fails to be a homeomorphism when some of the rectangles R_{ijk} have nonempty intersection.

We shall construct probability measures $\mu_{\mathbf{P},\mathbf{i}}$ supported on $\Lambda_{\mathbf{i}}$ with

$$\dim_{\mathrm{H}} \mu_{\mathbf{P},\mathbf{i}} = \lambda(\mathbf{P}) + t(\mathbf{P}) \quad \text{for } \mathbf{p}\text{-a.e. } \mathbf{i}.$$

This gives what we want because $\dim_{\mathrm{H}} \Lambda_{\mathbf{i}} \geq \dim_{\mathrm{H}} \mu_{\mathbf{P},\mathbf{i}}$.

Let $\tilde{\mu}_{\mathbf{P},\mathbf{i}}$ be the Bernoulli measure on $\Omega_{\mathbf{i}}$ such that

$$\tilde{\mu}_{\mathbf{P},\mathbf{i}}(C^{\mathbf{i}}_{\omega(n)}) = \prod_{l=1}^n \frac{p_{i_l j_l}}{p_{i_l}} \frac{a^{t(\mathbf{P})}_{i_l j_l k_l}}{\sum_k a^{t(\mathbf{P})}_{i_l j_l k}}.$$

Let $\mu_{\mathbf{P},\mathbf{i}}$ be the probability measure on $\Lambda_{\mathbf{i}}$ which is the pushforward of $\tilde{\mu}_{\mathbf{P},\mathbf{i}}$ by $\chi_{\mathbf{i}}$, i.e. $\mu_{\mathbf{P},\mathbf{i}} = \tilde{\mu}_{\mathbf{P},\mathbf{i}} \circ \chi_{\mathbf{i}}^{-1}$.

For calculating the Hausdorff dimension of $\mu_{\mathbf{P},\mathbf{i}}$ we shall consider some special sets called *approximate squares*. Given $\omega \in \Omega_{\mathbf{i}}$ and $n \in \mathbb{N}$ such that $n \geq (\log \min a_{ijk})/(\log \max b_{ij})$, define

$$L_n(\omega) = \max\left\{ k \geq 1 : \prod_{l=1}^n b_{i_l j_l} \leq \prod_{l=1}^k a_{i_l j_l k_l} \right\} \tag{5}$$

and the *approximate square*

$$B_n(\omega) = \{\omega' \in \Omega_{\mathbf{i}} : \ j'_l = j_l,\ l = 1, ..., n \text{ and } k'_l = k_l,\ l = 1, ..., L_n(\omega)\}.$$

We have that each approximate square $B_n(\omega)$ is a finite union of cylinder sets, and that approximate squares are *nested*, i.e., given two, say $B_n(\omega)$ and $B_{n'}(\omega')$, either $B_n(\omega) \cap B_{n'}(\omega') = \emptyset$ or $B_n(\omega) \subset B_{n'}(\omega')$ or $B_{n'}(\omega') \subset B_n(\omega)$. Moreover, $\chi_{\mathbf{i}}(B_n(\omega)) = \tilde{B}_n(\omega) \cap \Lambda_{\mathbf{i}}$ where $\tilde{B}_n(\omega)$ is a closed rectangle in $\mathbb{R}^2$ with edges parallel to the coordinate axes, with vertical length $\prod_{l=1}^n b_{i_l j_l}$ and horizontal length $\prod_{l=1}^{L_n(\omega)} a_{i_l j_l k_l}$. By (5),

$$1 \leq \frac{\prod\limits_{l=1}^{L_n(\omega)} a_{i_l j_l k_l}}{\prod\limits_{l=1}^{n} b_{i_l j_l}} \leq \max a^{-1}_{ijk}, \tag{6}$$

hence the term "approximate square". It follows from (6) that

$$\frac{\sum_{l=1}^{L_n(\omega)} \log a_{i_l j_l k_l}}{\sum_{l=1}^n \log b_{i_l j_l}} = 1 + \frac{1}{n} \frac{\sum_{l=1}^{L_n(\omega)} \log a_{i_l j_l k_l} - \sum_{l=1}^n \log b_{i_l j_l}}{\frac{1}{n}\sum_{l=1}^n \log b_{i_l j_l}} \to 1. \tag{7}$$

Also observe that $L_n(\omega) \leq n$ and $L_n(\omega) \to \infty$ as $n \to \infty$.

Lemma 1. $\dim_{\mathrm{H}} \mu_{\mathbf{P},\mathbf{i}} = \lambda(\mathbf{P}) + t(\mathbf{P})$ *for* $\mathbf{p}$*-a.e.* $\mathbf{i}$.

Proof. To calculate the Hausdorff dimension of $\mu_{\mathbf{P},\mathbf{i}}$ we are going calculate its pointwise dimension and use Proposition 1. Remember that $\chi_{\mathbf{i}}(B_n(\omega)) = \tilde{B}_n(\omega) \cap \Lambda_{\mathbf{i}}$ where, by (6), $\tilde{B}_n(\omega)$ is "approximately" a ball in $\mathbb{R}^2$ with radius $\prod_{l=1}^n b_{i_l j_l}$, and that

$$\mu_{\mathbf{P},\mathbf{i}}(\tilde{B}_n(\omega)) = \tilde{\mu}_{\mathbf{P},\mathbf{i}}(B_n(\omega)).$$

Also, $\chi_{\mathbf{i}}$ is at most 4 to 1. Taking this into account, by Proposition 1 together with [11, Theorem 15.3], one is left to prove that

$$\lim_{n\to\infty} \frac{\log \tilde{\mu}_{\mathbf{P},\mathbf{i}}(B_n(\omega))}{\sum_{l=1}^{n} \log b_{i_l j_l}} = \lambda(\mathbf{P}) + t(\mathbf{P}) \text{ for } \tilde{\mu}_{\mathbf{P},\mathbf{i}}\text{-a.e. } \omega \text{ and } \mathbf{p}\text{-a.e. } \mathbf{i}.$$

It follows from the definition of $\tilde{\mu}_{\mathbf{P},\mathbf{i}}$ that, for $\tilde{\mu}_{\mathbf{P},\mathbf{i}}$-a.e ω, $p_{i_l j_l} > 0$ for every l, so we may restrict our attention to these ω. We have that

$$\tilde{\mu}_{\mathbf{P},\mathbf{i}}(B_n(\omega)) = \prod_{l=1}^{n} \frac{p_{i_l j_l}}{p_{i_l}} \prod_{l=1}^{L_n(\omega)} \frac{a_{i_l j_l k_l}^{t(\mathbf{P})}}{\sum_k a_{i_l j_l k}^{t(\mathbf{P})}}$$

and

$$\begin{aligned}\frac{\log \tilde{\mu}_{\mathbf{P},\mathbf{i}}(B_n(\omega))}{\sum_{l=1}^{n} \log b_{i_l j_l}} &= \frac{\sum_{l=1}^{n} \log \frac{p_{i_l j_l}}{p_{i_l}}}{\sum_{l=1}^{n} \log b_{i_l j_l}} + t(\mathbf{P}) \frac{\sum_{l=1}^{L_n(\omega)} \log a_{i_l j_l k_l}}{\sum_{l=1}^{n} \log b_{i_l j_l}} \\ &\quad - \frac{\frac{1}{L_n(\omega)} \sum_{l=1}^{L_n(\omega)} \log\Big(\sum_k a_{i_l j_l k}^{t(\mathbf{P})} \Big)}{\frac{n}{L_n(\omega)} \frac{1}{n} \sum_{l=1}^{n} \log b_{i_l j_l}} \\ &= \frac{\alpha_n}{\beta_n} + t(\mathbf{P})\, \gamma_n - \frac{\delta_n}{\theta_n}.\end{aligned}$$

That $\gamma_n \to 1$ follows from (7). Now we can write

$$\alpha_n = \sum_{i,j} \frac{P(\omega, n, (i,j))}{n} \log \frac{p_{ij}}{p_i},$$

where

$$P(\omega, n, (i,j)) = \sharp\{1 \le l \le n : (i_l, j_l) = (i,j)\}.$$

By Kolmogorov's Strong Law of Large Numbers (KSLLN),

$$\frac{P(\omega, n, (i,j))}{n} \to p_{ij} \text{ for } \mathbf{P}\text{-a.e. } \omega,$$

so

$$\alpha_n \to \sum_{i,j} p_{ij} \log p_{ij} - \sum_i p_i \log p_i \text{ for } \mathbf{P}\text{-a.e. } \omega.$$

In the same way,

$$\beta_n \to \sum_{i,j} p_{ij} \log b_{ij} \text{ for } \mathbf{P}\text{-a.e. } \omega,$$

and, by the definition of $t(\mathbf{P})$,

$$\delta_n \to 0 \text{ for } \mathbf{P}\text{-a.e. } \omega.$$

Since $n/L_n^{d-1}(\omega) \ge 1$, we have that $|\theta_n| \ge \log(\min b_{ij}^{-1}) > 0$, so we also have that

$$\frac{\delta_n}{\theta_n} \to 0 \text{ for } \mathbf{P}\text{-a.e. } \omega,$$

thus completing the proof. □

As noticed in the beginning of this Part, these lemmas imply

$$\dim_{\mathrm{H}} \Lambda_{\mathbf{i}} \ge \sup_{\mathbf{P}} \{\lambda(\mathbf{P}) + t(\mathbf{P})\} \quad \text{for } \mathbf{p}\text{-a.e. } \mathbf{i}.$$

Part 2: $\dim_{\mathrm{H}} \Lambda_{\mathbf{i}} \le \sup_{\mathbf{P}} \{\lambda(\mathbf{P}) + t(\mathbf{P})\}$

Let $\underline{t} = \min_{\mathbf{P}} t(\mathbf{P})$ and $\overline{t} = \max_{\mathbf{P}} t(\mathbf{P})$. Also let $\mathcal{P}$ be the space of probability vectors $\mathbf{P} = (p_{ij})$ as before (projecting onto $\mathbf{p}$) such that $p_{ij} > 0$ for all (i,j).

Lemma 2. *Given $t \in (\underline{t}, \overline{t})$, there exists a probability vector $\mathbf{P} = \mathbf{P}(t) \in \mathcal{P}$, continuously varying, such that $t(\mathbf{P}) = t$ and*

$$p_{ij} = p_i\, b_{ij}^{\lambda(\mathbf{P})} \Big(\sum_k a_{ijk}^t\Big)^{\alpha} \Big(\sum_j b_{ij}^{\lambda(\mathbf{P})} \Big(\sum_k a_{ijk}^t\Big)^{\alpha}\Big)^{-1},$$

where $\alpha = \alpha(t) \in \mathbb{R}$ is C^1. Moreover, $d\alpha/dt > 0$ whenever $\alpha \in [0,1]$, and $\alpha(t) \to -\infty$ when $t \to \underline{t}$ and $\alpha(t) \to \infty$ when $t \to \overline{t}$.

Proof. Given $\alpha, \lambda \in \mathbb{R}$ and $t \in (\underline{t}, \overline{t})$, we define a probability vector $\mathbf{P}(\alpha, \lambda, t) \in \mathcal{P}$ by

$$p_{ij}(\alpha, \lambda, t) = p_i\, b_{ij}^{\lambda} \Big(\sum_k a_{ijk}^t\Big)^{\alpha} \gamma_i(\alpha, \lambda, t)^{-1} \tag{8}$$

where

$$\gamma_i(\alpha, \lambda, t) = \sum_j b_{ij}^{\lambda} \Big(\sum_k a_{ijk}^t\Big)^{\alpha}.$$

Let F be the continuous function defined by

$$F(\alpha, \lambda, t) = \sum_{i,j} p_{ij}(\alpha, \lambda, t) \log\Big(\sum_k a_{ijk}^t\Big). \tag{9}$$

We are going to prove there exists a unique $\alpha = \alpha(\lambda, t)$, continuously varying, such that $F(\alpha, \lambda, t) = 0$, i.e. $t(\mathbf{P}(\alpha, \lambda, t)) = t$.

Uniqueness. We have that,

$$\frac{\partial p_{ij}}{\partial \alpha} = \log\Big(\sum_j a_{ijk}^t\Big) p_{ij} - \frac{1}{\gamma_i}\frac{\partial \gamma_i}{\partial \alpha} p_{ij}.$$

Also,

$$\frac{1}{\gamma_i}\frac{\partial \gamma_i}{\partial \alpha} = \sum_j \frac{p_{ij}}{p_i} \log\Big(\sum_k a_{ijk}^t\Big). \tag{10}$$

So,

$$\begin{aligned}\frac{\partial F}{\partial \alpha} &= \sum_{i,j} \frac{\partial p_{ij}}{\partial \alpha} \log\Big(\sum_k a_{ijk}^t\Big) \\ &= \sum_i p_i \left\{ \sum_j \frac{p_{ij}}{p_i} \Big(\log\Big(\sum_k a_{ijk}^t\Big)\Big)^2 - \Big(\sum_j \frac{p_{ij}}{p_i} \log\Big(\sum_k a_{ijk}^t\Big)\Big)^2 \right\}. \end{aligned} \tag{11}$$

By the Cauchy-Schwarz inequality we have that the expression between curly brackets is non-negative and and is positive if there exists $i \in \{1, ..., m\}$ such that the function

$$j \mapsto \sum_k a_{ijk}^t$$

is non-constant (note that $\mathbf{P} \in \mathcal{P}$). This is guaranteed by hypothesis (1). Thus $\partial F/\partial \alpha > 0$.

Existence. For fixed (λ, t), we will look at the limit distributions of $\mathbf{P}(\alpha) = \mathbf{P}(\alpha, \lambda, t)$ as α goes to $+\infty$ and $-\infty$. We see that $\underline{t}$ and $\overline{t}$ are the unique solutions of the following equations, respectively,

$$\begin{aligned} &\sum_i p_i \log\Big(\min_j \sum_k a_{ijk}^t\Big) = 0 \\ &\sum_i p_i \log\Big(\max_j \sum_k a_{ijk}^t\Big) = 0. \end{aligned} \tag{12}$$

For instance, if t_* is the solution of (12) then $t_* = t(\mathbf{P})$ where $p_{ij} = p_i \delta_{ij(i)}$ where the maximum appearing in (12) is attained at $j(i)$, so $\overline{t} \geq t_*$. On the other hand, for all $\mathbf{P}$

$$\sum_i p_i \sum_j \frac{p_{ij}}{p_i} \log\Bigl(\sum_k a_{ijk}^{t_*}\Bigr) \leq \sum p_i \log\Bigl(\max_j \sum_k a_{ijk}^{t_*}\Bigr) = 0$$

which implies that $t(\mathbf{P}) \leq t_*$, and so $\overline{t} \leq t_*$. Now, for $t \in (\underline{t}, \overline{t})$, let

$$A_{i,t} = \max_j \sum_k a_{ijk}^t.$$

Then

$$\sum_{i,j} p_{ij}(\alpha) \log\Bigl(\sum_k a_{ijk}^t\Bigr) \underset{\alpha\to\infty}{\longrightarrow} \sum_i p_i \log A_{i,t} > 0. \tag{13}$$

In the same way, defining

$$B_{i,t} = \min_j \sum_k a_{ijk}^t,$$

we have

$$\sum_{i,j} p_{ij}(\alpha) \log\Bigl(\sum_k a_{ijk}^t\Bigr) \underset{\alpha\to-\infty}{\longrightarrow} \sum_i p_i \log B_{i,t} < 0. \tag{14}$$

By (13), (14) and continuity, there exists $\alpha \in \mathbb{R}$ such that $F(\alpha, \lambda, t) = 0$. The continuity of $\alpha(\lambda, t)$ follows from the uniqueness part and the implicit function theorem. Actually, since $F(\alpha, \lambda, t)$ is continuously differentiable, so is $\alpha(\lambda, t)$. Observe that $t(\mathbf{P}) = \overline{t} \Rightarrow \mathbf{P} \in \partial\mathcal{P}$ (in this lemma we are assuming $\underline{t} < \overline{t}$), so since

$$t(\mathbf{P}(\alpha(\lambda, t))) \to \overline{t} \quad \text{when} \quad t \to \overline{t}$$

then

$$\mathbf{P}(\alpha(\lambda, t)) \to \partial\mathcal{P} \quad \text{when} \quad t \to \overline{t},$$

which implies

$$\alpha(\lambda, t) \to \infty \quad \text{when} \quad t \to \overline{t}$$

(this convergence is uniform in $\lambda \in [0, 1]$). In the same way we see that

$$\alpha(\lambda, t) \to -\infty \quad \text{when} \quad t \to \underline{t}.$$

We use the following notation $\theta(\lambda, t) = (\alpha(\lambda, t), \lambda, t)$. We see that

$$\lambda(\mathbf{P}(\theta)) = \lambda - \frac{\sum_i p_i \log \gamma_i(\theta)}{\sum_{i,j} p_{ij} \log b_{ij}}.$$

So, we are left to prove there exists $\lambda = \lambda(t)$, continuously varying, such that

$$G(\theta) = \sum_i p_i \log \gamma_i(\theta) = 0. \tag{15}$$

We have that

$$\begin{aligned}
\frac{\partial}{\partial\lambda} \sum_i p_i \log \gamma_i(\theta) &= \sum_i p_i \frac{1}{\gamma_i(\theta)} \frac{\partial}{\partial\lambda} \gamma_i(\theta) \\
&= \sum_i p_i \left(\frac{1}{\gamma_i(\theta)} \frac{\partial\gamma_i}{\partial\alpha}(\theta) \frac{\partial\alpha}{\partial\lambda} + \frac{1}{\gamma_i(\theta)} \frac{\partial\gamma_i}{\partial\lambda}(\theta) \right) \\
&= \sum_i p_i \frac{1}{\gamma_i(\theta)} \frac{\partial\gamma_i}{\partial\lambda}(\theta)
\end{aligned}$$

where we have used (10). Now

$$\frac{1}{\gamma_i(\theta)} \frac{\partial\gamma_i}{\partial\lambda}(\theta) = \sum_j \frac{p_{ij}(\theta)}{p_i} \log b_{ij}, \tag{16}$$

so

$$\frac{\partial}{\partial\lambda}\sum_i p_i\log\gamma_i(\theta)=\sum_{i,j}p_{ij}(\theta)\log b_{ij}\le\max_{i,j}\log b_{ij}<0,$$

and, as before, by the implicit function theorem, this implies there exists a unique $\lambda=\lambda(t)$, continuously varying, satisfying (15).

To conclude the proof we must see that $d\alpha/dt>0$ whenever $\alpha\in[0,1]$. We have

$$\frac{d\alpha}{dt}=\frac{\partial\alpha}{\partial t}+\frac{\partial\alpha}{\partial\lambda}\frac{\partial\lambda}{\partial t}$$

Remember the definition of F from (9). Then

$$\frac{\partial\alpha}{\partial t}=-\left(\frac{\partial F}{\partial\alpha}\right)^{-1}\frac{\partial F}{\partial t}$$

where computation shows that

$$\begin{aligned}\frac{\partial F}{\partial t}&=\sum_{i,j}p_{ij}\frac{\sum_k a_{ijk}^t\log a_{ijk}}{\sum_k a_{ijk}^t} \qquad (17)\\ &+\alpha\sum_i p_i\Bigg\{\sum_j\frac{p_{ij}}{p_i}\frac{\sum_k a_{ijk}^t\log a_{ijk}}{\sum_k a_{ijk}^t}\log\Big(\sum_k a_{ijk}^t\Big)\\ &\qquad-\left(\sum_j\frac{p_{ij}}{p_i}\frac{\sum_k a_{ijk}^t\log a_{ijk}}{\sum_k a_{ijk}^t}\right)\left(\sum_j\frac{p_{ij}}{p_i}\log\Big(\sum_k a_{ijk}^t\Big)\right)\Bigg\}.\end{aligned}$$

Also

$$\frac{\partial\alpha}{\partial\lambda}=-\left(\frac{\partial F}{\partial\alpha}\right)^{-1}\frac{\partial F}{\partial\lambda}$$

and

$$\begin{aligned}\frac{\partial F}{\partial\lambda}=\sum_i p_i\Bigg\{&\sum_j\frac{p_{ij}}{p_i}\log b_{ij}\log\Big(\sum_k a_{ijk}^t\Big) \qquad (18)\\ &-\left(\sum_j\frac{p_{ij}}{p_i}\log b_{ij}\right)\left(\sum_j\frac{p_{ij}}{p_i}\log\Big(\sum_k a_{ijk}^t\Big)\right)\Bigg\}.\end{aligned}$$

It follows from (15) that

$$\frac{\partial\lambda}{\partial t}=-\left(\frac{\partial G}{\partial\lambda}\right)^{-1}\frac{\partial G}{\partial t}=-\alpha\frac{\sum_{i,j}p_{ij}\frac{\sum_k a_{ijk}^t\log a_{ijk}}{\sum_k a_{ijk}^t}}{\sum_{i,j}p_{ij}\log b_{ij}}.$$

Now using hypothesis (2), in the limit case when $\varepsilon\to 0$ we get

$$\frac{\partial F}{\partial t}\to\sum_i p_i\log a_i<0$$

which implies $\partial\alpha/\partial t>0$ (remember that $\partial F/\partial\alpha>0$). Also $\partial F/\partial\lambda$ and thus $\partial\alpha/\partial\lambda$ goes to 0, and $\partial\lambda/\partial t$ is bounded. This implies $d\alpha/dt>0$ which still holds if $\varepsilon>0$ is small enough. Now using hypothesis (3), in the limit case when $\varepsilon\to 0$ we get

$$\frac{d\alpha}{dt}\to-\left(\frac{\partial F}{\partial\alpha}\right)^{-1}\sum_{i,j}p_{ij}\log b_{ij}>0,$$

which still holds if $\varepsilon>0$ is small enough. □

Remark 2. We note that hypotheses (1)-(3) were only used in the previous lemma. Namely, hypothesis (1) was used to prove that $\partial F/\partial\alpha > 0$, see (11), and hypothesis (2) or (3) was used to prove that $d\alpha/dt > 0$, see (17) and (18). Do we really need these hypotheses?

Let $s = \sup_{\mathbf{P}}\{\lambda(\mathbf{P}) + t(\mathbf{P})\}$.

Lemma 3. *For* $\mathbf{p}$*-a.e.* $\mathbf{i}$ *and for every* $\omega \in \Omega_{\mathbf{i}}$ *there exists* $\mathbf{P} \in \mathcal{P}$ *such that*

$$\liminf_{n\to\infty} \frac{\log \tilde{\mu}_{\mathbf{P},\mathbf{i}}(B_n(\omega))}{\sum_{l=1}^n \log b_{i_l j_l}} \le s.$$

Proof. Fix $\mathbf{i}$ and $\omega \in \Omega_{\mathbf{i}}$. We use the notation

$$d_{\mathbf{P},\mathbf{i},n}(\omega) = \frac{\log \tilde{\mu}_{\mathbf{P},\mathbf{i}}(B_n(\omega))}{\sum_{l=1}^n \log b_{i_l j_l}}.$$

Then it follows from the proof of Lemma 1 that, if $\mathbf{P} \in \mathcal{P}$,

$$\begin{aligned} d_{\mathbf{P},\mathbf{i},n}(\omega) = & \frac{\sum_{l=1}^n \log p_{i_l j_l} - \sum_{l=1}^n \log p_{i_l}}{\sum_{l=1}^n \log b_{i_l j_l}} \\ & + \eta_n(\omega) t(\mathbf{P}) - \frac{\sum_{l=1}^{L_n(\omega)} \log\Big(\sum_k a_{i_l j_l k}^{t(\mathbf{P})}\Big)}{\sum_{l=1}^n \log b_{i_l j_l}} \end{aligned} \tag{19}$$

where

$$\eta_n(\omega) = \frac{\sum_{l=1}^{L_n(\omega)} \log a_{i_l j_l k_l}}{\sum_{l=1}^n \log b_{i_l j l}} \underset{n\to\infty}{\longrightarrow} 1.$$

Given $t \in (\underline{t}, \overline{t})$, consider the probability vector $\mathbf{P}(t)$, such that $t(\mathbf{P}(t)) = t$, given by Lemma 2. Applying (19) to $\mathbf{P}(t)$ we obtain

$$\begin{aligned} d_{\mathbf{P}(t),\mathbf{i},n}(\omega) = \lambda(\mathbf{P}(t)) + \eta_n(\omega) t - \frac{\sum_{l=1}^n \log \gamma_{i_l}(t)}{\sum_{l=1}^n \log b_{i_l j_l}} \\ + \frac{\alpha(t)\sum_{l=1}^n \log\Big(\sum_k a_{i_l j_l k}^t\Big) - \sum_{l=1}^{L_n(\omega)} \log\Big(\sum_k a_{i_l j_l k}^t\Big)}{\sum_{l=1}^n \log b_{i_l j_l}}, \end{aligned} \tag{20}$$

where, by KSLLN and (15),

$$\frac{1}{n}\sum_{l=1}^n \log \gamma_{i_l}(t) \to \sum_i p_i \log \gamma_i(t) = 0 \quad \text{for } \mathbf{p}\text{-a.e. } \mathbf{i}.$$

So we must prove that there exists $t_* \in (\underline{t}, \overline{t})$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\Big\{\alpha(t_*) \sum_{l=1}^n \log\Big(\sum_k a_{i_l j_l k}^{t_*}\Big) - \sum_{l=1}^{L_n(\omega)} \log\Big(\sum_k a_{i_l j_l k}^{t_*}\Big)\Big\} \ge 0. \tag{21}$$

By Lemma 2 and the inverse function theorem, given $a \in [0,1]$, there exists a unique function $t(a) \in (\underline{t}, \overline{t})$, which is continuous, increasing in a and satisfies

$$\alpha(t(a)) = a. \tag{22}$$

Let

$$a_0 = \liminf_{n\to\infty} \frac{L_n(\omega)}{n}, \quad a_1 = \limsup_{n\to\infty} \frac{L_n(\omega)}{n},$$

and

$$t_0 = t(a_0), \quad t_1 = t(a_1).$$

Let

$$a_n = \frac{L_n(\omega)}{n} \quad \text{and} \quad t_n = t(a_n).$$

Then, by Lemma 5 (and Remark 3), for every $t \in [t_0, t_1]$

$$F(t) = \limsup_{n\to\infty} \frac{1}{n}\Big\{a_n \sum_{l=1}^{n} \log\Big(\sum_k a_{i_l j_l k}^t\Big) - \sum_{l=1}^{L_n(\omega)} \log\Big(\sum_k a_{i_l j_l k}^t\Big)\Big\} \geq 0. \tag{23}$$

Since, for all (i,j),

$$t \mapsto \log\Big(\sum_k a_{ijk}^t\Big) \tag{24}$$

are continuous functions, so is $F(t)$. By adding some constant, we may assume the functions in (24) are ≥ 1, because by definition of a_n this does not change $F(t)$. Note that, by (22), $a_n = \alpha(t_n)$, and let $\bar{t}(t)$ be the biggest accumulation point of (t_n) for which the lim sup in (23) is attained. The continuity of F and the functions in (24) imply that $\bar{t}(t)$ is also continuous. So

$$F(t) = \limsup_{n\to\infty} \frac{1}{n}\Big\{\alpha(\bar{t}(t)) \sum_{l=1}^{n} \log\Big(\sum_k a_{i_l j_l k}^t\Big) - \sum_{l=1}^{L_n(\omega)} \log\Big(\sum_k a_{i_l j_l k}^t\Big)\Big\}.$$

Since

$$\bar{t}\colon [t_0, t_1] \to [t_0, t_1]$$

is continuous, by Brouwer's fixed point theorem there is $t_* \in [t_0, t_1]$ such that $\bar{t}(t_*) = t_*$, thus proving (21). □

Part 2 will be concluded in the following lemma.

Lemma 4. *For* $\mathbf{p}$*-a.e.* $\mathbf{i}$,

$$\dim_{\mathrm{H}} \Lambda_{\mathbf{i}} \leq \sup_{\mathbf{P}}\{\lambda(\mathbf{P}) + t(\mathbf{P})\}.$$

Proof. Let $\mathbf{i}$ be as in Lemma 3. Let $\varepsilon > 0$. Consider the *approximate squares of order* n given by $B_n(z) = \chi_{\mathbf{i}}(B_n(\omega))$ where $\omega \in \chi_{\mathbf{i}}^{-1}(z)$, $z \in \Lambda_{\mathbf{i}}$, $n \in \mathbb{N}$. Then it follows from Lemma 3 that

$$\forall_{z\in\Lambda_{\mathbf{i}}}\ \forall_{N\in\mathbb{N}}\ \exists_{n>N}\ \exists_{\mathbf{P}\in\mathcal{P}} : \ \frac{\log \mu_{\mathbf{P},\mathbf{i}}(B_n(z))}{\log |B_n(z)|} \leq s + \varepsilon. \tag{25}$$

Given $\delta, \eta > 0$, we shall build a cover $\mathcal{U}_{\delta,\eta}$ of $\Lambda_{\mathbf{i}}$ by sets with diameter $< \eta$ such that

$$\sum_{U\in\mathcal{U}_{\delta,\eta}} |U|^{s+\varepsilon+2\delta} \leq \sqrt{2}\, \max a_{ijk}^{-1}\, M_\delta$$

where M_δ is an integer depending on δ but not on η. This implies that $\dim_{\mathrm{H}} \Lambda_{\mathbf{i}} \leq s+\varepsilon+2\delta$ which gives what we want because ε and δ can be taken arbitrarily small. Let $b = \max b_{ij} < 1$. It is clear, using compactness arguments and the continuity of $\mathbf{P} \mapsto t(\mathbf{P})$, that there exists a finite number of Bernoulli measures $\mu_1, ..., \mu_{M_\delta}$ such that

$$\forall_{\mathbf{P}}\ \exists_{k\in\{1,\ldots,M_\delta\}} : \ \frac{\mu_{\mathbf{P},\mathbf{i}}(B_n)}{\mu_k(B_n)} \leq b^{-\delta n}$$

for all approximate squares of order n, B_n. By (25), we can build a cover of $\Lambda_{\mathbf{i}}$ by approximate squares $B_{n(z^l)}$, $l = 1, 2, ...$ that are disjoint and have diameters $< \eta$, such that

$$\mu_{\mathbf{P}^l,\mathbf{i}}(B_{n(z^l)}) \geq |B_{n(z^l)}|^{s+\varepsilon+\delta}$$

for some probability vectors $\mathbf{P}^l$. It follows that

$$\begin{aligned}\sum_l |B_{n(z^l)}|^{s+\varepsilon+2\delta} &\le \sum_l \mu_{\mathbf{P}^l,\mathbf{i}}(B_{n(z^l)})\,|B_{n(z^l)}|^{\delta}\\ &\le \sum_l \mu_{k_l}(B_{n(z^l)})\,b^{-\delta n(z^l)}\,\sqrt{2}\,\max a_{ijk}^{-1}\,b^{\delta n(z^l)}\\ &\le \sqrt{2}\,\max a_{ijk}^{-1}\sum_{k=1}^{M_\delta}\sum_l \mu_k(B_{n(z^l)}) \le \sqrt{2}\,\max a_{ijk}^{-1}\,M_\delta\end{aligned}$$

as we wish. □

This ends the proof of Theorem A.

4. A calculus lemma

Lemma 5. *Let $f\colon (0,\infty)\to\mathbb{R}$ be a Lipschitz function and $\alpha\colon (0,\infty)\to\mathbb{R}$ a positive bounded function. Then*

$$\limsup_{u\to\infty}\frac{1}{u}\Big(\alpha(u)f(u)-f(\alpha(u)u)\Big)\ge 0. \tag{26}$$

Proof. Let $g\colon (0,\infty)\to\mathbb{R}$ be defined by $g(x)=e^{-x}f(e^x)$. Then g is bounded and we must see that

$$\limsup_{x\to\infty}\alpha(e^x)\Big(g(x)-g(x+\log\alpha(e^x))\Big)\ge 0. \tag{27}$$

Just take a sequence $x_n\to\infty$ such that

$$\limsup_{x\to\infty} g(x)=\lim_{n\to\infty} g(x_n).$$

□

Remark 3. Lemma 5 also works when the functions f and α are defined only on the positive integers, by extending them in a piecewise linear fashion, if we substitute f being Lipschitz by

$$|f(n+1)-f(n)|\le C$$

for all n, for some constant $C>0$.

In what follows we make some extensions of Lemma 5 which are not used in this paper but should be useful when one tries to extend this paper to higher dimensions (see the problem proposed at the end of this section).

The next lemma is a non-linear extension of [7, Lemma 4.1] which was used to compute the Hausdorff dimension of multidimensional versions of general Sierpiński carpets.

Lemma 6. *Let $f_k\colon (0,\infty)\to\mathbb{R}$ be Lipschitz functions for $k=1,2,...,r$, and suppose $\alpha_k\colon (0,\infty)\to\mathbb{R}$ is bounded,* C^1 *and there exist positive constants δ, C such that*

$$\bullet\ \alpha_k(u)>\delta,\ \textit{for } u>0 \tag{28}$$

$$\bullet\ |u\,\alpha_k'(u)|\to 0\ \textit{as } u\to\infty. \tag{29}$$

Then

$$\limsup_{u\to\infty}\frac{1}{u}\sum_{k=1}^{r}\Big(\alpha_k(u)f_k(u)-f_k(\alpha_k(u)u)\Big)\ge 0. \tag{30}$$

Proof. As before and following [7], we define $g_k\colon (0,\infty) \to \mathbb{R}$ by $g_k(x) = e^{-x} f_k(e^x)$ for $k = 1, ..., r$. Then we must see that

$$\limsup_{x\to\infty} \sum_{k=1}^{r} \alpha_k(e^x)\Big(g_k(x) - g_k(x + \log \alpha_k(e^x))\Big) \geq 0. \tag{31}$$

We will see that

$$\left| \int_u^{b(u)} \sum_{k=1}^{r} \alpha_k(e^x)\Big(g_k(x) - g_k(x + \log \alpha_k(e^x))\Big)\, dx \right| \tag{32}$$

is bounded in u, for some $b(u) > u$ with $b(u) - u \to \infty$ as $u \to \infty$, which implies (31).

Let $\xi(u)$ be a decreasing function converging to 0 as $u \to \infty$ such that $|u\, \alpha_k'(u)| \leq \xi(u)$ for $u > 0$ ($\xi(u) = \max\{|x\, \alpha_k'(x)| : x \geq u\}$ will do). Then using (29) and

$$b(u) = u + \xi(e^u)^{-1}$$

we get that

$$\int_u^{b(u)} |\alpha_k'(e^x)\, e^x| \leq 1. \tag{33}$$

Note that the functions g_k are bounded because the functions f_k are Lipschitz. Using the intermediate value theorem and (33) we obtain

$$\left| \sum_{k=1}^{r} \int_u^{b(u)} \Big(\alpha_k(e^x) - \alpha_k(e^{x+\log \alpha_k(e^x)})\Big) g_k(x + \log \alpha_k(e^x))\, dx \right| \leq M$$

for $u > 0$ and some $M > 0$. So, (32) is bounded by M plus

$$\left| \int_u^{b(u)} \sum_{k=1}^{r} \Big(\alpha_k(e^x) g_k(x) - \alpha_k(e^{x+\log \alpha_k(e^x)}) g_k(x + \log \alpha_k(e^x))\Big)\, dx \right|. \tag{34}$$

By doing the change of coordinates $y = x + \log \alpha_k(e^x)$, which is invertible for $x > a$, for some $a > 0$, due to conditions (28) and (29), (34) becomes

$$\begin{aligned}
&\Bigg| \sum_{k=1}^{r} \Bigg(\int_u^{b(u)} \alpha_k(e^x) g_k(x)\, dx \\
&\qquad - \int_{u+\log \alpha_k(e^u)}^{b(u)+\log \alpha_k(e^{b(u)})} \alpha_k(e^y) g_k(y) \frac{\alpha_k(e^{x(y)})}{\alpha_k(e^{x(y)}) + \alpha_k'(e^{x(y)}) e^{x(y)}}\, dy \Bigg) \Bigg| \\
&\leq \sum_{k=1}^{r} \left| \int_u^{u+\log \alpha_k(e^u)} \alpha_k(e^x) g_k(x)\, dx \right| + \sum_{k=1}^{r} \left| \int_{b(u)+\log \alpha_k(e^{b(u)})}^{b(u)} \alpha_k(e^x) g_k(x)\, dx \right| \qquad (35) \\
&+ \sum_{k=1}^{r} \left| \int_{u+\log \alpha_k(e^u)}^{b(u)+\log \alpha_k(e^{b(u)})} \alpha_k(e^y) g_k(y) \frac{\alpha_k'(e^{x(y)}) e^{x(y)}}{\alpha_k(e^{x(y)}) + \alpha_k'(e^{x(y)}) e^{x(y)}}\, dy \right|. \qquad (36)
\end{aligned}$$

The terms in (35) are bounded because the functions g_k and α_k are bounded. That (36) is also bounded follows from (33). Thus (32) is bounded, concluding the proof. □

Corollary 1. *Suppose in addition to Lemma 6 hypotheses there are functions $\beta_k\colon (0,\infty) \to \mathbb{R}$, $k = 1, ..., r$ satisfying the same hypotheses of α_k. Then*

$$\limsup_{u\to\infty} \frac{1}{u} \sum_{k=1}^{r} \Big(\frac{\alpha_k(u)}{\beta_k(u)} f_k(\beta_k(u)u) - f_k(\alpha_k(u)u) \Big) \geq 0.$$

Proof. Define $g_k\colon (0,\infty) \to \mathbb{R}$ by $g_k(u) = f_k(\beta_k(u)u)$. Since the functions β_k satisfy the same conditions (28) and (29) as α_k, we can easily see that the functions $\frac{\alpha_k}{\beta_k}$ also do satisfy them, and that g_k are Lipschitz functions. Since

$$\frac{\alpha_k(u)}{\beta_k(u)} f_k(\beta_k(u)u) - f_k(\alpha_k(u)u) = \frac{\alpha_k(u)}{\beta_k(u)} g_k(u) - g_k\Big(\frac{\alpha_k(u)}{\beta_k(u)}u\Big),$$

we can apply Lemma 6. □

Unfortunately, Lemma 6 does not hold when we substitute hypothesis (29) by the weaker one

$$|u\,\alpha_k'(u)| \le C \tag{37}$$

for $u > 0$, for some constant $C > 0$. This is shown in the next example which was kindly communicated to me by Gustavo Moreira (Gugu).

Example 2. Let $h(x)$ and $c(x)$ be bounded C^1 functions with bounded derivative and $c(x) > b$ for all $x > 0$, for some $b > 0$. Then $f(x) = xh(\log x)$ is a Lipschitz function and $\alpha(x) = c(\log x)$ satisfies hypotheses (28) and (37). We have that

$$\alpha(u)f(u) - f(\alpha(u)u) = uc(\log u)(h(\log u) - h(\log u + \log c(\log u))).$$

If we put $h(x) = \sin x$ and $c(x) = e^{\cos x}$, we have that $\alpha(u)f(u) - f(\alpha(u)u) \le 0$ for all $u > 0$. Taking $h_k(x) = h(x + k\pi/2)$, $c_k(x) = c(x + k\pi/2)$, $f_k(x) = xh_k(\log x)$ and $\alpha_k(x) = c_k(\log x)$ for $k = 1, 2$, we have that

$$\frac{1}{u}\sum_{k=1}^{r}\Big(\alpha_k(u)f_k(u) - f_k(\alpha_k(u)u)\Big) \le -\delta$$

for all $u > 0$, for some $\delta > 0$.

Problem: Does Lemma 6 hold with hypothesis (37) instead of (29), for *generic* f_k *and* α_k ?

If the answer to this problem is affirmative in some sense then we believe we can compute the Hausdorff dimension of *generic self–affine Sierpiński sponges* which are the 3-dimensional versions of the self–affine Sierpiński carpets.

Acknowledgments: I wish to thank Gustavo Moreira for communicating to me Example 2. This work was supported by Fundação para a Ciência e a Tecnologia (Portugal).

References

[1] K. Barański, Hausdorff dimension of the limit sets of some planar geometric constructions, *Adv. Math.* **210** (2007), 215-245.
[2] T. Bedford, Crinkly curves, Markov partitions and box dimension of self similar sets, *PhD Thesis*, University of Warwick, 1984.
[3] R. Bowen, Hausdorff dimension of quasi-circles, *Publ. Math. I.H.E.S.* **50** (1979), 11-26.
[4] K. Falconer, *Fractal geometry, Mathematical foundations and applications*, Wiley, 1990.
[5] D. Gatzouras and P. Lalley, Hausdorff and box dimensions of certain self-affine fractals, *Indiana Univ. Math. J.* **41** (1992), 533-568.
[6] D. Gatzouras and Y. Peres, Invariant measures of full dimension for some expanding maps, *Ergod. Th. and Dynam. Sys.* **17** (1997), 147-167.
[7] R. Kenyon and Y. Peres, Measures of full dimension on affine-invariant sets, *Ergod. Th. & Dynam. Sys.* **16** (1996), 307-323.
[8] N. Luzia, A variational principle for the dimension for a class of non-conformal repellers, *Ergod. Th. & Dynam. Sys.* **26** (2006), 821-845.
[9] N. Luzia, Measure of full dimension for some nonconformal repellers, to appear in *Disc. & Cont. Dynam. Syst.* (http://www.arxiv.org/abs/0705.3604v1).

[10] C. McMullen, The Hausdorff dimension of general Sierpiński carpets, *Nagoya Math. J.* **96** (1984), 1-9.

[11] Ya. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago University Press, 1997.

[12] D. Ruelle, Repellers for real analytic maps, *Ergod. Th. and Dynam. Sys.* **2** (1982), 99-107.